§ 2.2 Clifford Operations

Operation U which takes Pauli product $[\cdots]$ to $\underbrace{U[\cdots]U^\dagger}_{\text{again Pauli product}}$ → "Clifford oper."

Consider the action of the Clifford operation U on the stabilizer state $|\psi\rangle$ defined by a stabilizer group $\mathcal{S} = \langle \{S_i\} \rangle$:

$$U|\psi\rangle = US_i|\psi\rangle = US_iU^\dagger U|\psi\rangle = S_i' U|\psi\rangle$$

where $S_i' \equiv U S_i U^\dagger$

$\Rightarrow$ $U|\psi\rangle$ is eigenvector of S_i' with eigenvalue $+1$ $\forall\ S_i'$

Heisenberg picture $\qquad\qquad$ Schrödinger picture

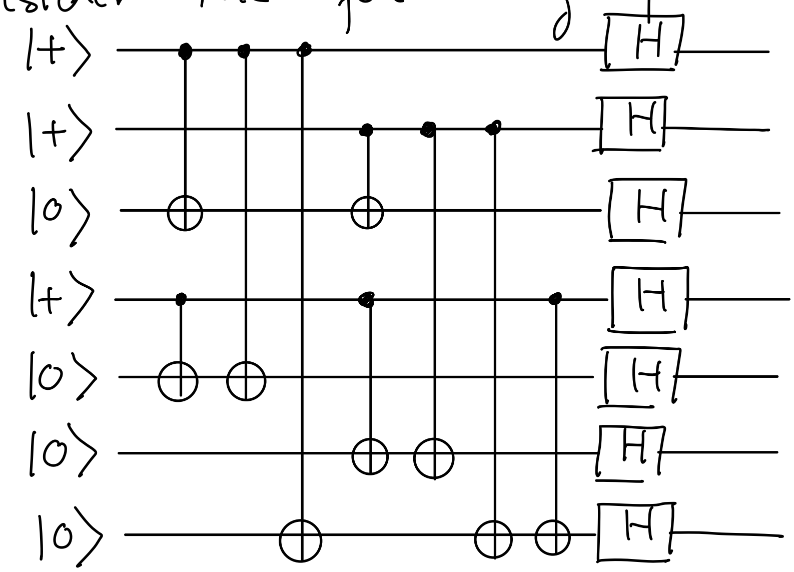
$$\begin{tikzcd}
\langle S_i \rangle \arrow[rr, leftrightarrow] \arrow[d, "U"'] & & |\Psi\rangle \arrow[d, "U"] \\
S_i' = US_iU^\dagger \ \langle S_i' \rangle \arrow[rr, leftrightarrow, "S_i'|\Psi\rangle = |\Psi'\rangle"'] & & |\Psi'\rangle = U|\Psi\rangle
\end{tikzcd}$$

Example 1:

the state stabilized by $\langle X_1 I_2, I_1 X_2 \rangle$ is $|+\rangle_1 |0\rangle_2$ $\longrightarrow$ stabilizer group is transformed under $\Lambda(X)_{1,2}$ into $\langle X_1 X_2, Z_1 Z_2 \rangle$ whose stabilizer state is $(|00\rangle + |11\rangle)/\sqrt{2}$

Example 2:

Consider the following quantum circuit



A calculation gives the following output state:

$$|\psi\rangle = \big(|0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle + |0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle + |1111111\rangle + |0101010\rangle + |1001100\rangle + |0011001\rangle + |1110000\rangle + |0100101\rangle + |1000110\rangle + |0010110\rangle\big)/4$$

Alternatively, we can understand the output state as stabilizer state of set

$$\{ZIZIZIZ,\ IZZIIZZ,\ IIIZZZZ,\ XXXIIII,\ XXIXXII,\ IXIXIXI,\ XIIXIIX\}$$

or alternatively the set

$$\{ZIZIZIZ,\ IZZIIZZ,\ IIIZZZZ,\ XXXXXXX,\ IIIXXXX,\ XIXIXIX,\ IXXIXXI\}$$

$\rightarrow$ $|\psi\rangle$ can be obtained from these by:

$$|\psi\rangle = 4\, \frac{I+S_4}{4}\, \frac{I+S_3}{2}\, \frac{I+S_2}{2}\, \frac{I+S_1}{2}\, |0000000\rangle$$

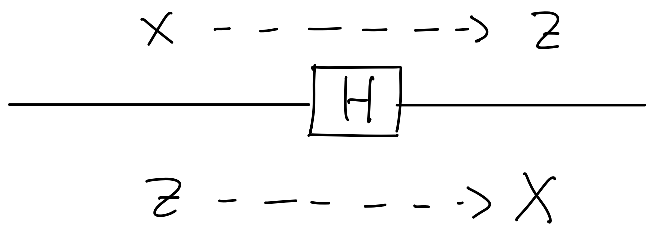
where $S_1 = XIXIXIX$, $S_2 = IXXIIXX$, $S_3 = IIIXXXX$, and $S_4 = XXXXXXX$

$|0000000\rangle$ is already an eigenstate of Z-stabilizers (with eigenvalue $+1$) and $\frac{I+S_i}{2}$ are projection operators onto the other stabilizer eigenstates

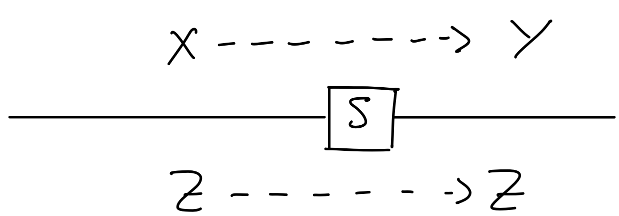
How do we obtain stabilizer generators of output state?

⟶ introduce commutation rules between Pauli and Clifford operation

1) $HX = ZH$ and $ZH = HX$



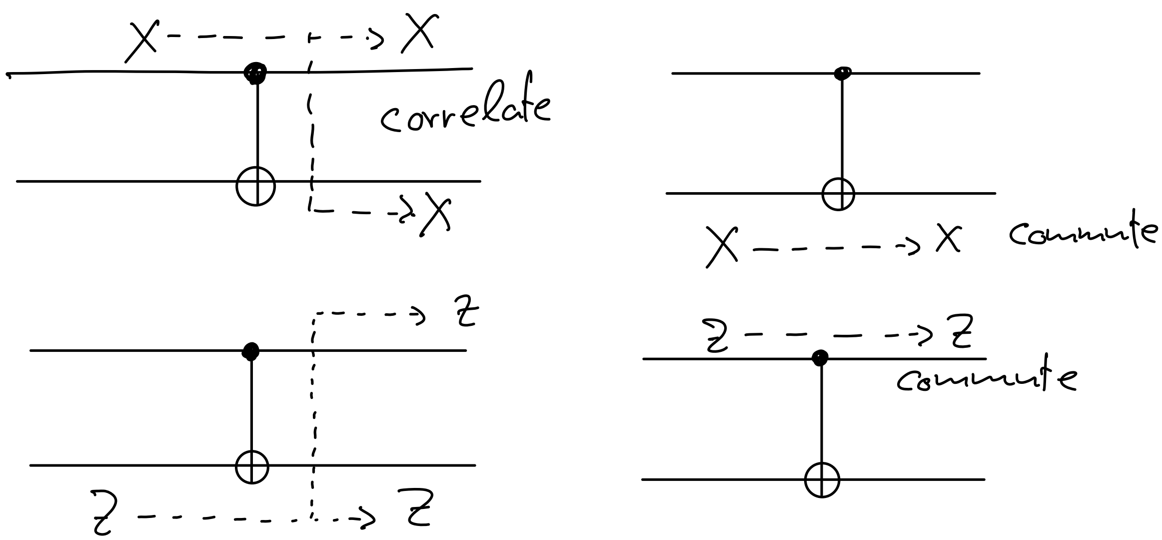
2) Similarly, for the phase operation S we have



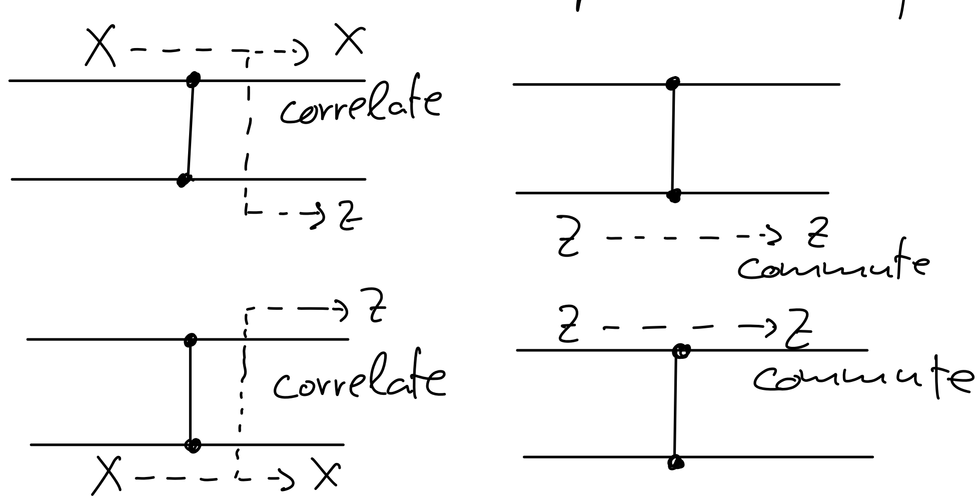
3) The CNOT operation transforms Pauli operators as follows:

$$\Lambda_{c,t}(X)\, X_c\, \Lambda_{c,t}(X) = X_c X_t$$
$$\Lambda_{c,t}(X)\, X_t\, \Lambda_{c,t}(X) = X_t,$$
$$\Lambda_{c,t}(X)\, Z_c\, \Lambda_{c,t}(X) = Z_c,$$
$$\Lambda_{c,t}(X)\, Z_t\, \Lambda_{c,t}(X) = Z_c Z_t$$

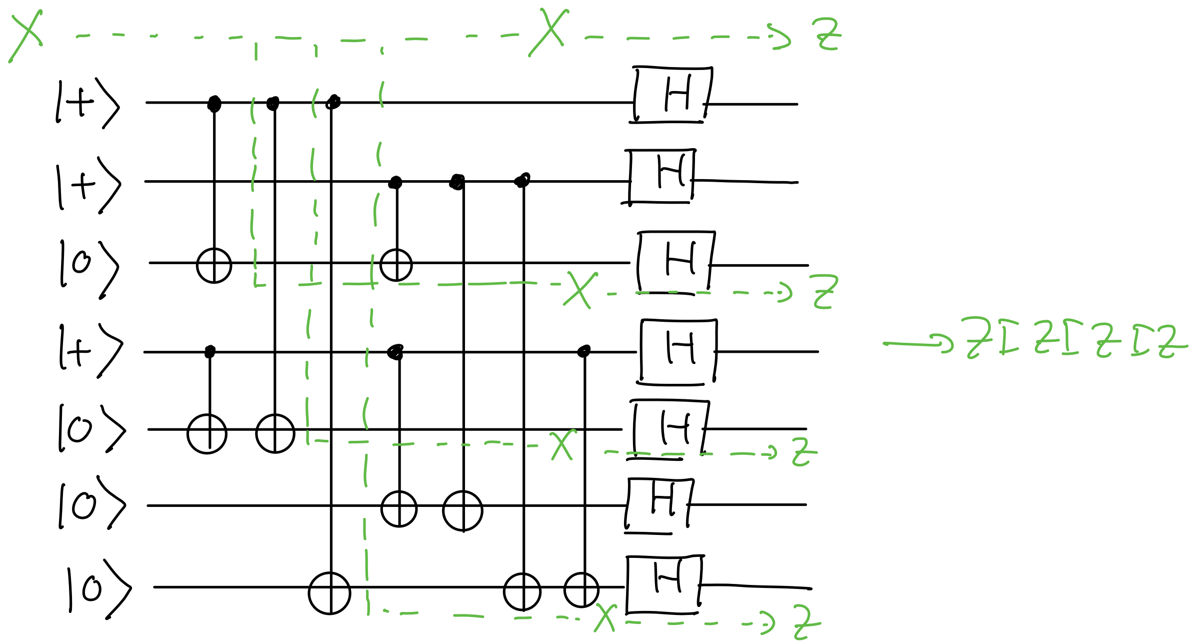
or in pictures:



4) The CZ operation commutes with the Pauli operators as follows:



The above commutation relations can be used to compute the stabilizers of the output for our circuit:



Other stabilizer elements can be computed analogously.

§2.3 Pauli Basis Measurements

Suppose the A-basis ($A = X, Y, Z$) measurement is performed on a stabilizer state $|\psi\rangle$ with stabilizer group $\langle S_i \rangle$.

Assume $\#\{S_i\} = \#$ qubit $\rightarrow$ quantum state can be pinned down exactly

→ two possibilities:

i) Pauli op. A commutes with all stabilizer generators

→ either A or $-A \in \langle S_i \rangle$

→ eigenvalue + (-1) is obtained with probability 1

→ post-measurement state is same as before

ii) $\exists\ \tilde{S} \in \langle S_i \rangle : \ [\tilde{S}, A] \neq 0$

→ choose another set of generators $\{S_i'\}$ such that $\{S_1', A\} = 0$

bu $[S_j', A] = 0 \quad \forall\, j \geq 1$

measurement outcomes $(-1)^m$ lead to post-measurement set:

$\langle (-1)^m A,\ S_2', \ldots, S_k' \rangle$

<u>Example</u>: consider $\mathcal{S}_{Bell} = \langle XX, ZZ \rangle$

→ redefine to $\langle S_i' \rangle = \{XX, -YY\}$

→ after measurement: $\langle (-1)^m YI, -YY \rangle$

§2.4 Gottesman-Knill Theorem

Theorem 1:

Any Clifford operations, applied to the input state $|0\rangle^{\otimes n}$ followed by Z-measurements, can be simulated efficiently in the strong sense.

means: classical simulation of a quantum circuit C in polynomial time giving probability $P_C(x)$ of given output state x